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Orbital Covariance Interpolation

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This work derives two interpolators to determine the intermediate covariance of a space object's position. Two considerations are given to the derivations. The first is that the covariance matrix changes direction and shape with orbital motion as reflected in its first and second derivatives with respect to time. In the absence of such derivatives, the second consideration is that covariance growth can be reasonably modeled by including data that is outside the immediate time interval of concern. These two considerations incorporate orbital motion with time-associated covariance growth/reorientation to produce realistic intermediate covariance matrices while precisely matching them at the start and end of a given time interval. The method is computationally simple, using matrix algebra with no eigenvalue or eigenvector determination. Earth-Centered Inertial data (ECI) does not require a coordinate transformation for the first interpolator.

Introduction

An ephemeris file contains position and velocity information at discrete times along with their associated covariance matrices. Collision analysis for off-discrete times requires position and velocity interpolation[1,2]. Covariance data must also be interpolated for probability computations. A simple linear transformation (morphing) of the positional covariance elements from one time to the next may not be sufficient if the rotation associated with orbital motion and other elements is not taken into account. Such a linear transformation might match the covariance ellipsoid at the end points of a time interval while distorting the intermediate ellipsoid. An example would be to visualize a broomstick rotated 90 degrees. At the midpoint of the time interval, the intermediate ellipsoid should look like a broomstick rotated 45 degrees. If an improper transformation model (such as linear morphing) were used, then the intermediate ellipsoid would look like a circular disk; the axes would be shrinking/expanding rather than rotating. The same problem arises in attitude interpolation and has been successfully handled using 2-point osculating interpolation methods[3]. Similarly, after examining numerous alternate methods, a recommended approach to three-dimensional covariance visualization uses eigenvalue-eigenvector decomposition[4].

The purpose of this study is to derive and examine two covariance interpolators that account for orbital motion as well as growth and re-orientation without eigenvalue-eigenvector decomposition. This study is only concerned with

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propagated/predicted data and does not address measurement processing/updating. The first method uses the first and second covariance derivatives with respect to time and then splines the data as outlined in Reference 2. The second method splines only the positional covariance matrices.

Derivation

It is assumed that position (r), velocity (v), and their associated covariance data (P) are all in the Earth-Centered Inertial (ECI) frame. The time interval between data sets is normalized to one ($0 = \tau = 1$). The data are represented as r0, v0, P0 at the beginning of the time interval and r1, v1, P1 at the end. Because this derivation only involves first and second derivatives of non-linear motion, it is recommended that the time interval not exceed more than 15° of orbital motion. The time interval should be carefully chosen to satisfy user accuracy requirements. The positional quintic spline equation is defined as

$$p(\tau) = a0 + a1\cdot\tau + a2\cdot\tau^2 + a3\cdot\tau^3 + a4\cdot\tau^4 + a5\cdot\tau^5 \quad . \tag{1}$$

Given information of position (pa, pb), velocity (va, vb), and acceleration (aa, ab) at the beginning and end of a unitized time interval (0, 1) the coefficients can be computed from the equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{pmatrix} \cdot \begin{pmatrix} a0 \\ a1 \\ a2 \\ a3 \\ a4 \\ a5 \end{pmatrix} = \begin{pmatrix} pa \\ pb \\ va \\ vb \\ aa \\ ab \end{pmatrix} \quad . \tag{2}$$

Rearranging terms, the coefficients become

$$\begin{pmatrix} a0 \\ a1 \\ a2 \\ a3 \\ a4 \\ a5 \end{pmatrix} = \begin{pmatrix} pa \\ va \\ .5\cdot aa \\ -10.\cdot pa + 10.\cdot pb - 6.\cdot va - 4.\cdot vb - 1.5\cdot aa + .5\cdot ab \\ 15.\cdot pa - 15.\cdot pb + 8.\cdot va + 7.\cdot vb + 1.5\cdot aa - 1.\cdot ab \\ -6.\cdot pa + 6.\cdot pb - 3.\cdot va - 3.\cdot vb - .5\cdot aa + .5\cdot ab \end{pmatrix} \tag{3}$$

Substituting the coefficients of Equation 3 into Equation 1, it is useful to regroup the terms based on the interval data such that

$$\begin{aligned} p(\tau) = {} & \left[-\left(6\cdot\tau^2 + 3\cdot\tau + 1\right)\cdot(\tau - 1)^3\right]\cdot pa + \left[\tau^3\cdot\left(10 - 15\cdot\tau + 6\cdot\tau^2\right)\right]\cdot pb \\ & + \left[-\tau\cdot(3\cdot\tau + 1)\cdot(\tau - 1)^3\right]\cdot va + \left[-\tau^3\cdot(\tau - 1)\cdot(3\cdot\tau - 4)\right]\cdot vb \\ & + \left[\frac{-1}{2}\cdot\tau^2\cdot(\tau - 1)^3\right]\cdot aa + \left[\frac{1}{2}\cdot\tau^3\cdot(\tau - 1)^2\right]\cdot ab \quad . \end{aligned} \tag{4}$$

Because the nature of the interval data was never specified as an element, vector, or matrix, Equation 4 holds for all. The bracketed terms of Equation 4 are scalars and need only be computed once for a given τ. The same equation can then be used to determine an interpolated 3x1 position vector, 3x3 covariance matrix and/or a 6x6 covariance matrix.

The reader is cautioned that the time interval for Equation 4 is normalized, requiring the rates and accelerations to be scaled accordingly. As an example, if the units associated with rate were kilometers per second and the time interval between consecutive data points was 50 seconds, then the velocity data would be scaled by 50 to produce (va, vb). Acceleration data would be scaled by 50^2 to produce (aa, ab).

In the absence of user-provided covariance time derivatives for orbital motion, they can be approximated using a simple two-body Keplerian model[5]. The first time derivative of a 6x6 covariance P is given by

$$\frac{dP}{dt} = F\cdot P + P\cdot F^T \tag{5}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \tag{6}$$

and

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\mu}{r^3} + \frac{3\cdot\mu}{r^5}\cdot x^2 & \frac{3\cdot\mu}{r^5}\cdot x\cdot y & \frac{3\cdot\mu}{r^5}\cdot x\cdot z & 0 & 0 & 0 \\ \frac{3\cdot\mu}{r^5}\cdot x\cdot y & \frac{-\mu}{r^3} + \frac{3\cdot\mu}{r^5}\cdot y^2 & \frac{3\cdot\mu}{r^5}\cdot y\cdot z & 0 & 0 & 0 \\ \frac{3\cdot\mu}{r^5}\cdot x\cdot z & \frac{3\cdot\mu}{r^5}\cdot y\cdot z & \frac{-\mu}{r^3} + \frac{3\cdot\mu}{r}\cdot z^2 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

The x, y, z elements of the position vector determine the magnitude r. The gravitational parameter is μ. Given a zero-indexed 6x6 covariance matrix P, the upper 3x3 (positional) covariance derivative becomes

$$\frac{dP}{dt} = \begin{pmatrix} 2\cdot P_{0,3} & P_{1,3} + P_{0,4} & P_{2,3} + P_{0,5} \\ P_{1,3} + P_{0,4} & 2\cdot P_{1,4} & P_{2,4} + P_{1,5} \\ P_{2,3} + P_{0,5} & P_{2,4} + P_{1,5} & 2\cdot P_{2,5} \end{pmatrix} \tag{8}$$

The second time derivative of the 6x6 covariance P is given by

$$\frac{d^2P}{dt^2} = \frac{dF}{dt}\cdot P + P\cdot\frac{dF}{dt}^T + F\cdot\frac{dP}{dt} + \frac{dP}{dt}\cdot F^T \tag{9}$$

The upper diagonal elements of the symmetric 3x3 positional covariance second derivative become

$$\left(\frac{d^2P}{dt^2}\right)_{0,0} = 2\cdot\left(P_{3,3} + F_{3,0}\cdot P_{0,0} + F_{3,1}\cdot P_{0,1} + F_{3,2}\cdot P_{0,2}\right) \tag{10}$$

$$\left(\frac{d^2P}{dt^2}\right)_{0,1} = \left(F_{3,0} + F_{4,1}\right)\cdot P_{0,1} + \left(P_{0,0} + P_{1,1}\right)\cdot F_{3,1} + 2\cdot P_{3,4} + F_{4,2}\cdot P_{0,2} + F_{3,2}\cdot P_{1,2} \tag{11}$$

$$\left(\frac{d^2P}{dt^2}\right)_{0,2} = \left(F_{3,0} + F_{5,2}\right)\cdot P_{0,2} + \left(P_{2,2} + P_{0,0}\right)\cdot F_{3,2} + 2\cdot P_{3,5} + F_{3,1}\cdot P_{1,2} + F_{4,2}\cdot P_{0,1} \tag{12}$$

$$\left(\frac{d^2P}{dt^2}\right)_{1,1} = 2\cdot\left(P_{4,4} + F_{3,1}\cdot P_{0,1} + F_{4,1}\cdot P_{1,1} + F_{4,2}\cdot P_{1,2}\right) \tag{13}$$

$$\left(\frac{d^2P}{dt^2}\right)_{1,2} = \left(P_{2,2} + P_{1,1}\right)\cdot F_{4,2} + \left(F_{4,1} + F_{5,2}\right)\cdot P_{1,2} + 2\cdot P_{4,5} + F_{3,1}\cdot P_{0,2} + F_{3,2}\cdot P_{0,1} \tag{14}$$

$$\left(\frac{d^2P}{dt^2}\right)_{2,2} = 2\cdot\left(P_{5,5} + F_{3,2}\cdot P_{0,2} + F_{4,2}\cdot P_{1,2} + F_{5,2}\cdot P_{2,2}\right) \tag{15}$$

Alternate Derivation

If only the positional 3x3 covariance is available, then an alternate splining method is used. The unitized time interval of interest remains ($0 = \tau = 1$) but six consecutive data sets are required spanning ($-2 = \tau = 3$). The data/τ pairs are represented as (pa, -2), (pb, -1), (pc, 0), (pd, 1), (pe, 2), and (pf, 3). It is recommended that the time interval not exceed more than 10 degrees of orbital motion. Again, the time interval should be chosen to satisfy user accuracy requirements. The alternate positional quintic spline equation is

$$p(\tau) = b0 + b1\cdot\tau + b2\cdot\tau^2 + b3\cdot\tau^3 + b4\cdot\tau^4 + b5\cdot\tau^5 \quad . \tag{16}$$

The coefficients can be computed from the equation

$$\begin{pmatrix} 1 & -2 & 4 & -8 & 16 & -32 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \end{pmatrix} \cdot \begin{pmatrix} b0 \\ b1 \\ b2 \\ b3 \\ b4 \\ b5 \end{pmatrix} = \begin{pmatrix} pa \\ pb \\ pc \\ pd \\ pe \\ pf \end{pmatrix} \quad . \tag{17}$$

Rearranging terms, the coefficients become

$$\begin{pmatrix} b0 \\ b1 \\ b2 \\ b3 \\ b4 \\ b5 \end{pmatrix} = \begin{pmatrix} pc \\ \frac{1}{20}\cdot pa - \frac{1}{2}\cdot pb - \frac{1}{3}\cdot pc + pd - \frac{1}{4}\cdot pe + \frac{1}{30}\cdot pf \\ \frac{-1}{24}\cdot pa + \frac{2}{3}\cdot pb - \frac{5}{4}\cdot pc + \frac{2}{3}\cdot pd - \frac{1}{24}\cdot pe \\ \frac{-1}{24}\cdot pa - \frac{1}{24}\cdot pb + \frac{5}{12}\cdot pc - \frac{7}{12}\cdot pd + \frac{7}{24}\cdot pe - \frac{1}{24}\cdot pf \\ \frac{1}{24}\cdot pa - \frac{1}{6}\cdot pb + \frac{1}{4}\cdot pc - \frac{1}{6}\cdot pd + \frac{1}{24}\cdot pe \\ \frac{-1}{120}\cdot pa + \frac{1}{24}\cdot pb - \frac{1}{12}\cdot pc + \frac{1}{12}\cdot pd - \frac{1}{24}\cdot pe + \frac{1}{120}\cdot pf \end{pmatrix} \tag{18}$$

Substituting the coefficients of Equation 18 into Equation 16, it is useful to regroup the terms based on the positional data points

$$\begin{aligned} p(\tau) = & \left[\frac{-1}{120}\cdot\tau\cdot(\tau-1)\cdot(\tau-2)\cdot(\tau-3)\cdot(\tau+1)\right]\cdot pa + \left[\frac{1}{24}\cdot\tau\cdot(\tau-1)\cdot(\tau-2)\cdot(\tau-3)\cdot(\tau+2)\right]\cdot pb \\ & + \left[\frac{-1}{12}\cdot(\tau-1)\cdot(\tau-2)\cdot(\tau-3)\cdot(\tau+2)\cdot(\tau+1)\right]\cdot pc + \left[\frac{1}{12}\cdot\tau\cdot(\tau-2)\cdot(\tau-3)\cdot(\tau+2)\cdot(\tau+1)\right]\cdot pd \\ & + \left[\frac{-1}{24}\cdot\tau\cdot(\tau-1)\cdot(\tau-3)\cdot(\tau+2)\cdot(\tau+1)\right]\cdot pe + \left[\frac{1}{120}\cdot\tau\cdot(\tau-1)\cdot(\tau-2)\cdot(\tau+2)\cdot(\tau+1)\right]\cdot pf \end{aligned} \tag{19}$$

The above equation can be used to determine an interpolated 3x3 covariance matrix over the interval ($0 = \tau = 1$) using the six neighboring covariance matrices. Accuracy can be improved by rotating all matrices from the ECI frame to the Transverse-Normal-Other (TNW) frame where T is along the velocity vector, N is along the angular momentum vector, and W completes the right-hand system. Splining is performed in the TNW frame and the interpolated covariance matrix is then transformed back to the ECI frame.

Numerical Testing

The methods were tested for accuracy regarding geosynchronous (GEO), low-earth (LEO), Global Positioning System (GPS), and Molniya (MYA) orbits in the ECI frame. The primary method was tested only in the ECI frame; the alternate method was tested in both the ECI and TNW frames. The errors were smaller using the TNW frame, so only those results are shown. The interpolated covariance was compared to the actual by examining the angle and magnitude differences between ellipsoid axes.

TABLE 1. Primary Method Maximum Errors for GEO

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
15	0.11826	0.00054
30	0.12088	0.00058
45	0.17814	0.00079
60	0.60204	0.00219
75	1.95670	0.00713
90	5.84920	0.02072

TABLE 2. Primary Method Maximum Errors for LEO

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
2.0	0.21096	0.04481
3.0	0.29865	0.10066
4.0	0.33702	0.18040
5.0	0.35375	0.28967
6.0	0.39677	0.44146
7.0	0.54304	0.66012
8.0	0.72384	0.98600
9.0	0.95474	1.48032

TABLE 3. Primary Method Maximum Errors for GPS

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
5	0.19331	0.00246
10	0.17852	0.00291
15	0.18511	0.00239
20	0.20788	0.00279
25	0.38321	0.00607
30	0.85673	0.01382
35	1.73740	0.02744

TABLE 4. Primary Method Maximum Errors for MYA

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
2	0.38521	0.00311
4	0.38305	0.00368
6	0.41066	0.00393
8	0.40671	0.00393
10	0.43217	0.00393
12	0.42818	0.00394
14	0.92876	0.00394
16	2.05050	0.00861

Tables 1-4 show that the primary method's maximum error can be held below one percent if the change in true anomaly is less than or equal to 15°.

TABLE 5. Alternate Method Maximum Errors for GEO

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
15	0.1024	0.000555
30	0.255	0.0013661
45	1.8949897	0.0093176
60	9.9146719	0.05130088

TABLE 6. Alternate Method Maximum Errors for LEO

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
2.0	0.20792	0.03321
2.5	0.24517	0.04134
3.0	0.34983	0.07963
3.5	0.36042	0.18367
4.0	0.39241	0.39608
4.5	0.62742	0.79436
5.0	1.16595	1.48035

TABLE 7. Alternate Method Maximum Errors for GPS

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
5	0.09494	0.0020859
10	0.10354	0.0016646
15	0.13112	0.0026841
20	0.32133	0.012792
25	0.95457	0.048255
30	2.7846	0.14123
35	6.7018	0.34573

TABLE 8. Alternate Method Maximum Errors for MYA

Interval (minutes)	Magnitude Error (%)	Angle Error (degrees)
2	0.42501	0.0032767
4	0.39319	0.0035228
6	0.4098	0.0039264
8	0.96829	0.0085885
10	3.5194	0.030092
12	10.0499	0.081608

As expected, the alternate method did not perform as well as the primary. Tables 5-8 show that the alternate method's maximum error can be held below one percent if the change in true anomaly is less than or equal to 10°. It is still useful when 3x3 covariance information is the only data available, albeit with smaller time intervals.

For the limited number of test cases, the interpolated 3x3 covariance matrices were always positive definite. An area for future study is to prove that positive definiteness is preserved with these methods.

Conclusion

Two covariance interpolators were derived for propagated ephemerides that do not involve eigenvalue or eigenvector computations. The primary method uses the first and second derivatives of the covariance matrix to produce a quintic spline. In the absence of such derivatives, the alternate method approximates covariance behavior in the TNW frame by including data that is outside the immediate time interval of concern. The methods use only simple matrix algebra. It remains to be proven that the interpolated covariance is always positive definite.

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